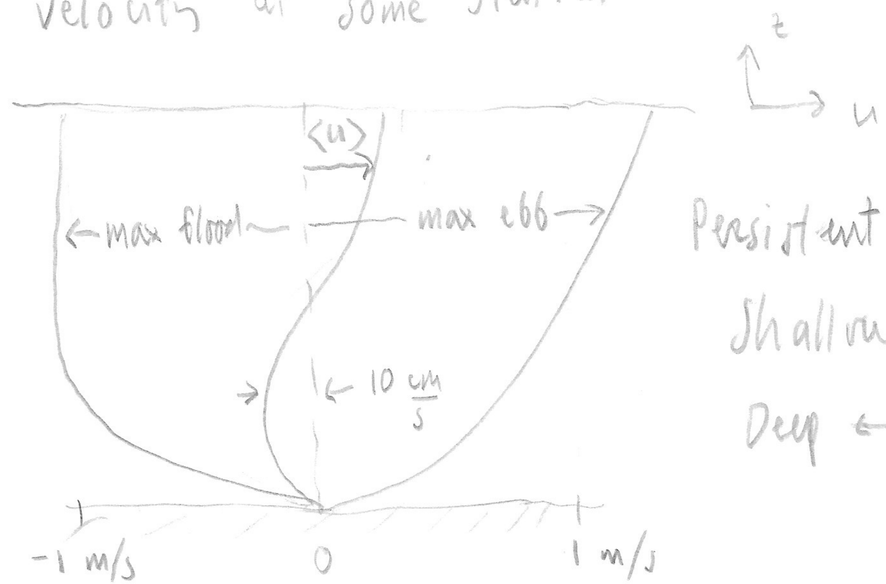
Estuarine Circulation

Estuary: a long bay influenced by rivers + tides

Observed salinity structure (examples in MacCready + Banas 2011)

- Vertically stratified
- Along-channel density gradient

Observed velocity at some station:



Persistent "exchange flow"

Shallow → out

Deep ← in

$\langle u \rangle \equiv$ tidally averaged u ("subtidal", "residual")

Tidal-averaging: e.g. $\langle u \rangle \big|_{t_0}$ = weighted average of 40-71 hours of u around t_0 (2)

e.g. LiveOcean/alpha/zfun.filt-godin()

- What causes exchange flow?

$$\langle \boxed{x \text{ mom}} \rangle \quad \underbrace{\cancel{\langle u \rangle_t}}_{\sim\text{steady exchange}} + \underbrace{\cancel{\langle \underline{u} \cdot \nabla u \rangle}}_{\text{hope advection is small}} - \underbrace{f\cancel{\langle v \rangle}}_{\text{narrow channel } \langle v \rangle \sim 0} = \underbrace{-\frac{1}{\rho_0}\langle p_x \rangle + \langle (A_v u_z)_z \rangle}_{\text{pressure gradient balanced by vertical stress divergence}}$$

Pressure: hydrostatic $\Rightarrow p = g\int_z^{\eta} \rho \, dz$ and $\rho = \rho_0(1+\beta s)$

$\beta = 7.7 \times 10^{-4}$

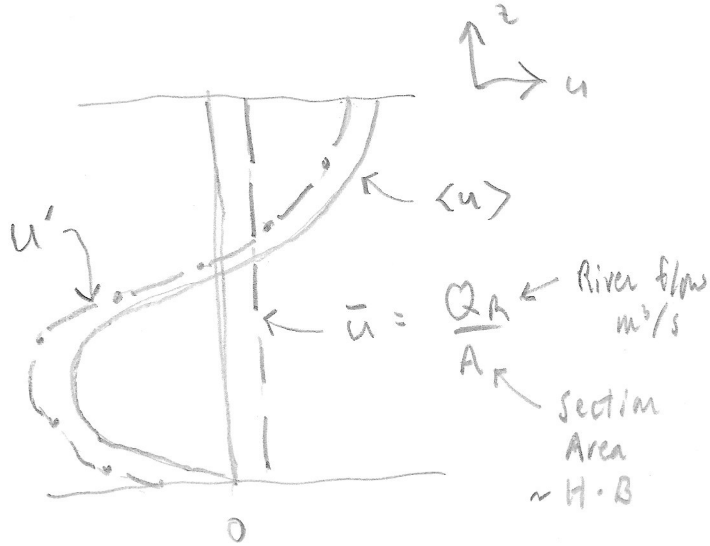
$\rho_0 = 1000 \ kg/m^3$

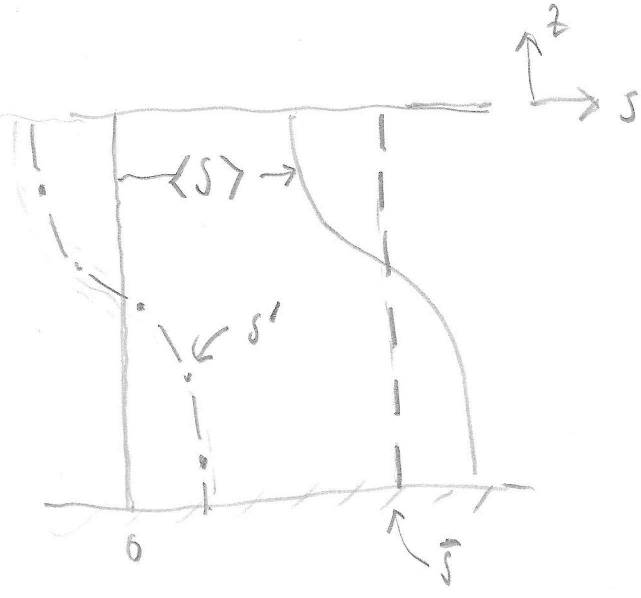
$$\Rightarrow p_x \cong \rho_0 g \eta_x + g\int_z^0 \rho_0 \beta s_x \, dz$$

and
$$-\frac{\langle p_x \rangle}{\rho_0} = -g\langle \eta_x \rangle - g\int_z^0 \beta \langle s_x \rangle \, dz$$

We are only concerned with tidally-averaged flow so drop the $\langle \; \rangle$ when writing variables.

And we decompose into vertically-averaged and depth-varying parts:



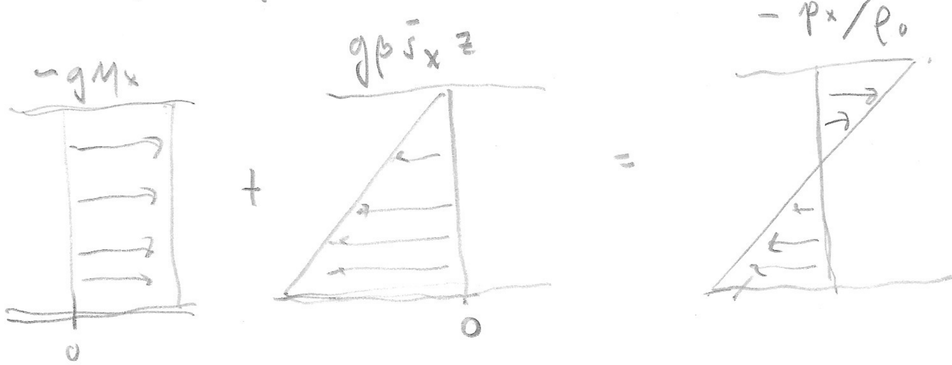


$\langle u \rangle = \bar{u} + u'$ $\qquad$ $\langle s \rangle = \bar{s} + s'$

We observe (Pritchard 1954): $[s'_x] \ll [\bar{s}_x]$

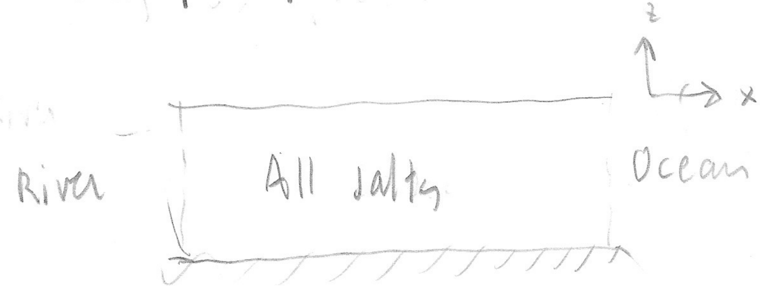
so $-\frac{p_x}{\rho_0} = -g\eta_x + g\beta \bar{s}_x z$ $\qquad$ (dropped $\langle\,\rangle$'s)

For η_x negative and $\bar{s}_x$ positive:

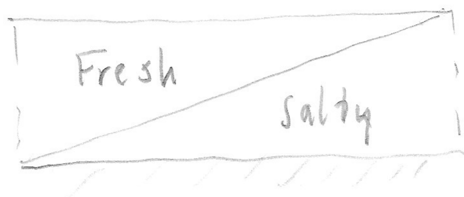


Forces exchange flow

Thought problem about $\bar{S}_x$:



1 Start all salty



2 Add Fresh water

3 Mix vertically (tides)

$\Rightarrow \bar{S}_x > 0$

Then
- Shallow exchange flow removes mixed water
- river adds FW
- deep exchange flow adds salt water

(back to 1/2)

In reality all are happening at once, or system is going $2 \longleftrightarrow 3$ over spring-neap cycle.

We are doing math to find $u(z)$ given $\bar{S}_x$

To find $u(z)$ we solve

assume A = const.

$$0 = -g\eta_x + g\beta\bar{S}_x z + A u_{zz}$$

subject to boundary conditions

(i) $u(-H) = 0$ No slip at bottom

(ii) $u_z(0) = 0$ No stress at top

and integral constraint:

(iii) $\frac{1}{H}\int_{-H}^{0} u \, dz = \bar{u}$ note: $\int_{-H}^{0} u' \, dz = 0$

A, $\bar{S}_x$ and $\bar{u}$ are assumed known

so we integrate twice in z, using (i) + (ii)

and use (iii) to find right value of η_x.

Result is cubic in z:

$$u = \bar{u} + \overbrace{\left[\bar{u}\,\tfrac{1}{2}(1 - 3\zeta^2) + u_E(1 - 9\zeta^2 - 8\zeta^3) \right]}^{u'}$$

where $\zeta = z/H$ and $u_E = \frac{g\beta\bar{S}_x H}{48} \frac{H^2}{A}$

u_E expresses balance: pressure gradient

vertical friction time scale

u_E is the scale of the exchange flow

Note $u_E \sim H^3$

$\Rightarrow$ exchange much stronger for deep estuaries ?

No: because $\bar{S}_x$ adjusts dynamically depending on how much mixing happens

So: we need to consider controls on $\langle s \rangle$.

(*) For the frictionless case, if we allow acceleration:

$$u_t = -g\eta_x + g\beta \bar{S}_x z$$

- If $\bar{S}_x = 30 / 50\text{ km}$, $H = 20\text{ m}$, and $\eta_x = 0$ how fast is the bottom water going after 3 hours ?

- what value of η_x would be required to oppose the acceleration due to $\bar{S}_x$ so that $\int_{-H}^{0} u_t \, dz = 0$?

My answers:

$$\frac{30}{10^5}\frac{1}{m} = \frac{\Delta S}{L}$$

- $u_t = -g\cancel{\eta_x} + g\beta \bar{S}_x z$

assume 0 ; $z \to -H$

$$\Rightarrow u = -g\beta\bar{S}_x H t = -\overset{1}{10}\cdot 7.7\times10^{-4}\cdot\frac{50}{10^5}\cdot\overset{1}{20}\cdot\overset{3}{3600}\cdot 3$$

$$= 7.7\times10^{-4}\cdot\underbrace{\underbrace{50\cdot 2}_{100}\cdot\underbrace{3.6\cdot 3}_{10}}_{10^3} = 0.8\ \text{m/s}$$

$$= u(-H, 3\text{ hours})$$

- $\int_{-H}^{0} u_t\, dz = 0 = -g\eta_x H + g\beta\bar{S}_x\left(\frac{1}{2}z^2\right)\Big|_{-H}^{0}$

$$\cancel{g}\eta_x \cancel{H} = -\frac{1}{2}\cancel{g}\beta\bar{S}_x H^{\cancel{2}}$$

$$\frac{\Delta\eta}{\cancel{L}} = -\frac{1}{2}\beta\frac{\Delta S}{\cancel{L}}H = -\frac{1}{2}7.7\times10^{-4}\times\overbrace{50\times20}^{10^3}$$

$$= -\frac{1}{2}7.7\times10^{-1} \approx -0.4\ \text{m}\ !$$

$$= \Delta\eta$$

But observations suggest η_x is much smaller than this.

Geyer et al. (2000 JPO) Hudson River Estuary

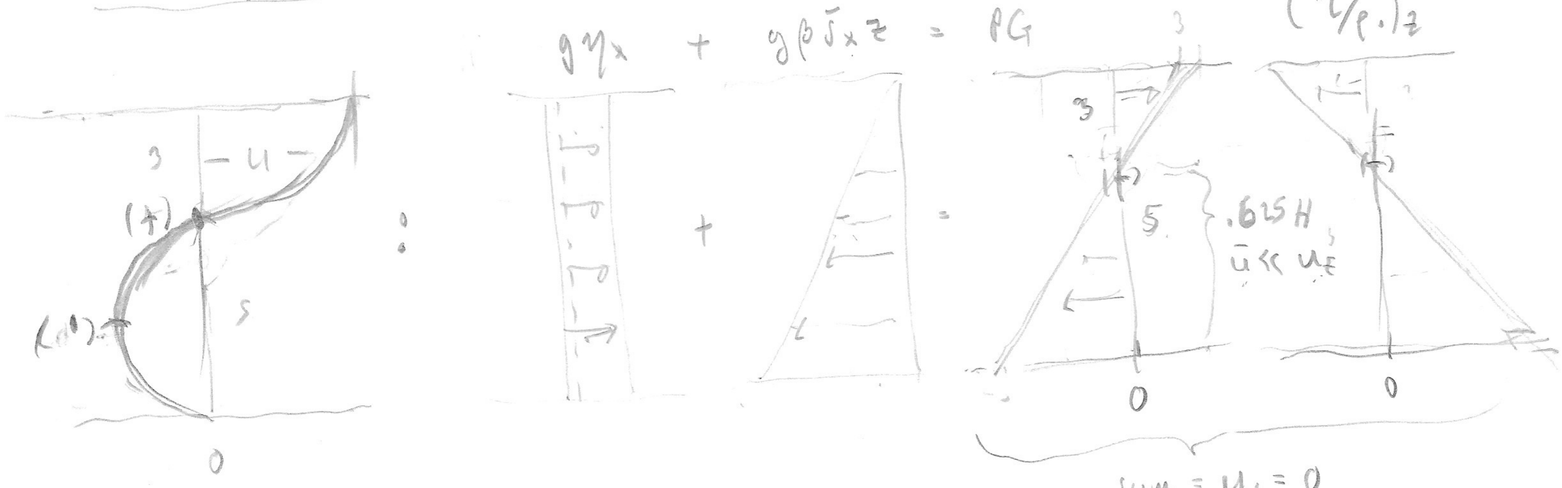
Looking at the force balance :

$\boxed{x \text{ mom}}$ $$\underbrace{u_t}_{\text{steady}} = \underbrace{-\frac{1}{\rho_0} p_x}_{PG} + \frac{\partial}{\partial z}\left(\tau/\rho_0\right), \quad \frac{\text{stress}}{\rho_0} = \frac{\tau}{\rho_0} = A\frac{\partial u}{\partial z} = -\langle u'w' \rangle$$

sign convention

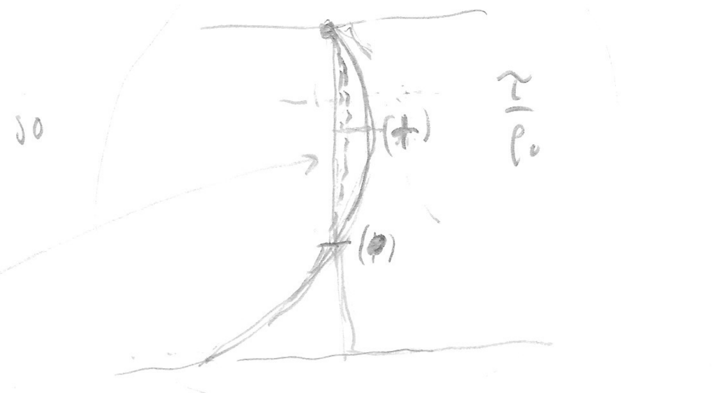
unidirectional:

exchange flow



In observations in Hudson River (Geyer et al. 2000 JGR)

the stress in the surface layer, and η_x, are much smaller ---



Appendix what is the Au_{zz} profile for our simple cubic u ?

8/6/2019

$$u_t = -g\eta_x + g\beta \bar{J}_x z + Au_{zz} \quad u_E$$

$$u = \bar{u} + \bar{u}\tfrac{1}{2}(1-3\zeta^2) + u_E(1-9\zeta^2-8\zeta^3)]$$

$\zeta = z/H$

$\partial\zeta = \frac{1}{H}\partial z$

$dz = H\partial\zeta$

$$\boxed{\frac{\partial}{\partial z} = \frac{1}{H}\frac{\partial}{\partial \zeta}}$$

$$u_z = \bar{u}(-3)2\frac{\zeta}{H} + u_E\left[(-9)(2)\frac{\zeta}{H} - 8(3)\frac{\zeta^2}{H}\right]$$

$$u_z = -6\bar{u}\frac{\zeta}{H} + u_E\left(-18\frac{\zeta}{H} - 24\frac{\zeta^2}{H}\right)$$

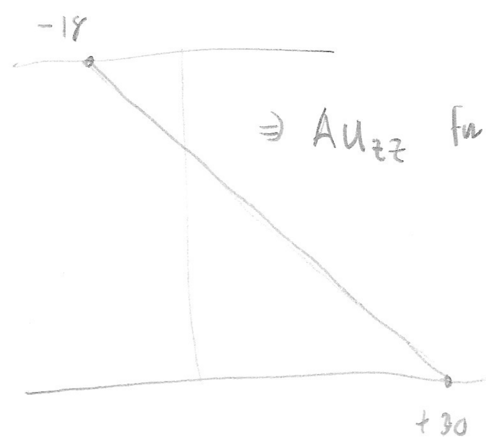
$$u_{zz} = -\frac{6\bar{u}}{H^2} + u_E\left(-\frac{18}{H^2} - \frac{48\zeta}{H^2}\right)$$

$$u_{zz}\Big|_0 = \frac{-6\bar{u}}{H^2} + u_E\left(\frac{-18}{H^2}\right)$$

$$u_{zz}\Big|_{-H} = \frac{-6\bar{u}}{H^2} + u_E\left(\frac{-18}{H^2} + \frac{48}{H^2}\right) = -\frac{6\bar{u}}{H^2} + u_E\frac{30}{H^2}$$

$\zeta = -1$

$$\boxed{\begin{aligned} Au_{zz}\Big|_0 &= \frac{A}{H^2}\left(-6\bar{u} - 18u_E\right) \\ A_{zz}\Big|_{-H} &= \frac{A}{H^2}\left(-6\bar{u} + 30u_E\right) \end{aligned}}$$



$\Rightarrow Au_{zz}$ for $\bar{u} \ll u_E$